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Short Communication

Analytical approximations to the double-well Duffing oscillator in large amplitude oscillations

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Abstract

Accurate analytical approximate solutions to the double-well Duffing oscillator are presented. The solutions are obtained by combining Newton's method with the harmonic balance method. The procedure yields rapid convergence with respect to exact solution. The results are valid for small as well as large oscillation amplitudes.

The Duffing equation with a double-well potential (with a negative linear stiffness) is an important model. One physical realization of such a Duffing oscillator model is a mass particle moving in a symmetric double well potential. This form of the equation also appears in the transverse vibrations of a beam when the transverse and longitudal deflections are coupled [1]. The damped and forced double-well Duffing equation has been a subject of intensive study over the last few decades as a landmark chaotic system, we refer readers to Ref. [2] and cited therein. Although the Duffing equation with a nonnegative linear stiffness is very often used as an example to demonstrate the validity of various methods for constructing analytical approximate solutions to nonlinear oscillators [3–5], no corresponding report to the Duffing oscillator with a negative linear stiffness appears up to now, to the authors' knowledge. This paper uses the methods proposed by Wu et al. [6–8] to construct analytical approximate periods and periodic solutions to free oscillation of the undamped double-well Duffing oscillator. The solutions are obtained by combining Newton's method with the harmonic balance method. The procedure yields rapid convergence with respect to exact solution. The results are valid for small as well as large oscillation amplitudes.

Consider a conservative single-degree-of-freedom system governed by

$$\frac{d^2u}{dt^2}+f(u)=0, \quad u(0)=A, \quad \frac{du}{dt}(0)=0. \tag{1}$$

For convenience, the approach presented by Wu et al. [6] for the case of $f(-u)=-f(u)$, is briefly summarized as follows. By coupling the Newton method with the method of harmonic balance, Wu et al. [6] obtained three analytical approximate periods and corresponding periodic solutions. The first analytical

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approximation to the period and periodic solution is

$$T_1(A) = 2\pi/\sqrt{\Omega_1(A)}, \quad u_1(t) = A\cos\tau, \quad \tau = \sqrt{\Omega_1(A)}t, \tag{2}$$

where $A>0$ represents oscillation amplitude and

$$\Omega_1(A) = a_1/A, \quad a_{2i-1} = \frac{4}{\pi}\int_0^{\pi/2} f(A\cos\tau)\cos[(2i-1)\tau]\,d\tau, \quad i = 1, 2, \ldots. \tag{3}$$

The second analytical approximation to the period and periodic solution is

$$\begin{aligned} &T_2(A) = 2\pi/\sqrt{\Omega_2(A)}, \quad \Omega_2(A) = \Omega_1(A) + \Delta\Omega_1(A), \\ &u_2(t) = X(A)\cos\tau + Y(A)\cos 3\tau, \quad \tau = \sqrt{\Omega_2(A)}t, \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\Delta\Omega_1(A) = a_3[2a_1 - (b_0 - b_4)A]/\Phi(A), \quad X(A) = A - 2a_3A^2/\Phi(A), \\ &Y(A) = 2a_3A^2/\Phi(A), \quad \Phi(A) = A[(b_2 + b_4 - b_0 - b_6)A + 18a_1], \\ &b_{2(i-1)} = \frac{4}{\pi}\int_0^{\pi/2} f_u(A\cos\tau)\cos[2(i-1)\tau]\,d\tau, \quad i = 1, 2, \ldots. \end{aligned} \tag{5}$$

The third analytical approximation to the period and periodic solution is

$$\begin{aligned} &T_3(A) = 2\pi/\sqrt{\Omega_3(A)}, \quad \Omega_3(A) = \Omega_2(A) + \Delta\Omega_2(A), \\ &u_3(t) = \left[X(A) + y_1(A)\right]\cos\tau + \left[Y(A) - y_1(A) + y_2(A)\right]\cos 3\tau - y_2(A)\cos 5\tau, \quad \tau = \sqrt{\Omega_3(A)}t, \end{aligned} \tag{6}$$

where $\Delta\Omega_2(A)$, $y_1(A)$, $y_2(A)$ can be obtained by solving a set of linear algebra equations. Here, they are omitted for saving space, and for details, we refer readers to Ref. [8].

For the case of $f(u)$ being a general nonlinear function of u, let $V(u) = \int f(u)\,du$ be the potential energy of the system and suppose it reach its minimum at $u = u_0$, called a center. We assume $u_0 = 0$. Thus, the system oscillates between asymmetric limits $[-B, A]$ where both $u = -B (B>0)$ and $u = A$ have the same energy level:

$$V(-B) = V(A). \tag{7}$$

Here, B and A represent the left and right oscillation amplitudes, respectively.

Following the approach in Ref. [7], we introduce the two odd nonlinear oscillating systems:

$$\frac{d^2u}{dt^2} + \mathrm{K}(u,\alpha) = 0, \quad u(0) = H, \quad \frac{du}{dt}(0) = 0, \tag{8}$$

where

$$\mathrm{K}(u,\alpha) \equiv \begin{cases} \alpha f(\alpha u) & \text{if } u \geqslant 0, \\ -\alpha f(-\alpha u) & \text{if } u < 0 \end{cases} \tag{9}$$

with $\alpha = \pm 1$. Here we set $H = A$ for $\alpha = 1$ and $H = B$ for $\alpha = -1$, respectively. In Eqs. (2)–(6), replacing f with $K(u,\alpha)$ for $\alpha = \pm 1$, respectively, we may achieve the corresponding first, second and third analytical approximate periods and the periodic solutions $T_n^{+1}(A)$, $u_n^{+1}(t)$ and $T_n^{-1}(B)$, $u_n^{-1}(t)$ ($n = 1,2,3$). Utilizing these analytical approximate solutions, we can construct the corresponding the nth ($n = 1,2,3$) analytical approximate period and periodic solution as follows [7]:

$$T_n(A) = \frac{T_n^{+1}(A)}{2} + \frac{T_n^{-1}(B)}{2} \tag{10a}$$

and

$$u_n(t) = \begin{cases} u_n^{+1}(t) & \text{for } 0 \leqslant t \leqslant \dfrac{T_n^{+1}(A)}{4}, \\ u_n^{-1}\left(t - \dfrac{T_n^{+1}(A)}{4} + \dfrac{T_n^{-1}(B)}{4}\right) & \text{for } \dfrac{T_n^{+1}(A)}{4} \leqslant t \leqslant \dfrac{T_n^{+1}(A)}{4} + \dfrac{T_n^{-1}(B)}{2}, \\ u_n^{+1}\left(t + \dfrac{T_n^{+1}(A)}{2} - \dfrac{T_n^{-1}(B)}{2}\right) & \text{for } \dfrac{T_n^{+1}(A)}{4} + \dfrac{T_n^{-1}(B)}{2} \leqslant t \leqslant \dfrac{T_n^{+1}(A)}{2} + \dfrac{T_n^{-1}(B)}{2}. \end{cases} \quad (10b)$$

Now, we study the unforced and undamped double-well Duffing oscillator:

$$\frac{d^2u}{dt^2} - u + u^3 = 0, \quad u(0) = A, \quad \frac{du}{dt}(0) = 0. \quad (11)$$

Potential energy of this system is given by $V(u) = -u^2/2 + u^4/4$ and it has three equilibrium points. The central equilibrium point $u = 0$ is unstable and the other two $u = \pm 1$ are stable. The periodic solutions of this system depend upon the initial oscillation amplitude A. For the case of $0 < A < 1$ and $1 < A < \sqrt{2}$, oscillation occurs around stable equilibrium point $u = \pm 1$, which is asymmetric about this point. For the case of $A > \sqrt{2}$, the periodic solution is a symmetric one and extends across three equilibrium points.

Oscillation for the case of $A > \sqrt{2}$ is first studied, which occurs between symmetric limits $[-A, A]$. For this case, one has $f(u) = -u + u^3$ and $f_u(u) = -1 + 3u^2$. The Fourier series expansions of $f(A\cos\tau)$ and $f_u(A\cos\tau)$ are given in Eqs. (3) and (5), respectively, where

$$a_1 = -A + 3A^3/4, \quad a_3 = A^3/4, \quad b_0 = -2 + 3A^2, \quad b_2 = 3A^2/2, \quad b_4 = b_6 = 0. \quad (12)$$

Using Eqs. (2)–(5) and (12), we obtain the first two analytical approximate formulas for the period and periodic solution:

$$T_1(A) = 2\pi\Big/\sqrt{\Omega_1(A)}, \quad \Omega_1(A) = 3A^2/4 - 1, \quad u_1(t) = A\cos\tau, \quad \tau = \sqrt{\Omega_1(A)}t \quad (13)$$

and

$$T_2(A) = 2\pi\Big/\sqrt{\Omega_2(A)}, \quad \Omega_2(A) = \frac{128 - 192A^2 + 69A^4}{96A^2 - 128},$$
$$u_2(t) = \left(\frac{32A - 23A^3}{32 - 24A^2}\right)\cos\tau + \left(\frac{A^3}{24A^2 - 32}\right)\cos 3\tau, \quad \tau = \sqrt{\Omega_2(A)}t. \quad (14)$$

Based on Eqs. (6) and (14), the third analytical approximate expressions for the period and periodic solution are:

$$T_3(A) = 2\pi/\sqrt{\Omega_3(A)}, \quad \Omega_3(A) = \frac{D(A)}{4L(A)},$$
$$u_3(t) = [L_1(A)\cos\tau + L_2(A)\cos 3\tau + L_3(A)(12A^2 - 16)\cos 5\tau]/L(A), \quad \tau = \sqrt{\Omega_3(A)}t, \quad (15)$$

where

$$\begin{aligned} D(A) = & -1{,}099{,}511{,}627{,}776 + 7{,}352{,}984{,}010{,}752A^2 - 21{,}769{,}041{,}739{,}776A^4 + 37{,}447{,}618{,}527{,}232A^6 \\ & - 41{,}248{,}951{,}894{,}016A^8 + 30{,}171{,}363{,}606{,}528A^{10} - 1{,}465{,}423{,}202{,}184A^{12} + 4{,}557{,}352{,}944{,}960A^{14} \\ & - 823{,}439{,}591{,}472A^{16} + 65{,}856{,}986{,}475A^{18}, \end{aligned}$$

$$
\begin{aligned}
L_1(A) = {} & 274{,}877{,}906{,}944A - 1{,}623{,}497{,}637{,}888A^3 + 4{,}179{,}808{,}485{,}376A^5 \\
& - 6{,}126{,}905{,}065{,}472A^7 + 5{,}592{,}752{,}848{,}896A^9 - 3{,}255{,}431{,}946{,}240A^{11} + 1{,}180{,}009{,}138{,}944A^{13} \\
& - 243{,}516{,}596{,}624A^{15} + 21{,}904{,}831{,}241A^{17}, \\
L_2(A) = {} & -8{,}589{,}934{,}592A^3 + 44{,}560{,}285{,}696A^5 - 98{,}750{,}693{,}376A^7 + 121{,}196{,}511{,}232A^9 \\
& - 88{,}969{,}502{,}720A^{11} + 39{,}067{,}159{,}296A^{13} - 9{,}501{,}566{,}864A^{15} + 987{,}420{,}271A^{17}, \\
L_3(A) = {} & -16{,}777{,}216A^5 + 61{,}865{,}984A^7 - 90{,}947{,}584A^9 + 66{,}641{,}920A^{11} \\
& - 24{,}344{,}256A^{13} + 3{,}547{,}175A^{15}, \\
L(A) = {} & 4(68{,}719{,}476{,}736 - 408{,}021{,}893{,}120A^2 + 1{,}056{,}159{,}301{,}632A^4 - 1{,}556{,}711{,}735{,}296A^6 \\
& + 1{,}429{,}036{,}728{,}320A^8 - 836{,}639{,}772{,}672A^{10} + 305{,}066{,}377{,}344A^{12} \\
& - 63{,}341{,}762{,}340A^{14} + 5{,}733{,}704{,}403A^{16}).
\end{aligned} \tag{16}
$$

The exact period $T_e(A)$ for Eq. (11) is

$$
T_e(A) = \int_0^{\pi/2} \frac{4\,\mathrm{d}t}{\sqrt{A^2(1+\sin^2 t)/2 - 1}}. \tag{17}
$$

For comparison, the exact period $T_e(A)$ obtained by computing integral in Eq. (17) and the approximate periods T_1, T_2, and T_3 computed, respectively, by Eqs. (13)–(15) are listed in Table 1. Furthermore, we have

$$
\lim_{A\to+\infty} \frac{T_1}{T_e} = 0.978277, \quad \lim_{A\to+\infty} \frac{T_2}{T_e} = 0.999318, \quad \lim_{A\to+\infty} \frac{T_3}{T_e} = 0.999929. \tag{18}
$$

Note that, for this oscillator, oscillation amplitude is required to satisfy $A > \sqrt{2}$, since Eq. (11) has a homoclinic orbit with period $+\infty$ for $A = \sqrt{2}$. From Table 1 and Eq. (18), it can be concluded that Eqs. (13)–(15) can provide excellent approximate periods for oscillation amplitude $A > \sqrt{2}$.

For purpose of comparison, the exact periodic solutions $u_e(t)$ achieved by integrating Eq. (11) and the analytical approximate periodic solutions $u_1(t)$, $u_2(t)$ and $u_3(t)$ computed by Eqs. (13)–(15), respectively, are plotted in Figs. 1 and 2 for the time in one period. These figures correspond to, respectively, two different amplitudes of oscillation $A = 1.5$ and 10.

Figs. 1 and 2 show that the third approximations provide the most excellent solutions with respect to the exact periodic solutions for oscillation amplitude $A > \sqrt{2}$. The proposed second approximations are generally acceptable for large oscillation amplitude.

For the case of $1 < A < \sqrt{2}$, the oscillation occurs around stable equilibrium points $u = +1$, and is a asymmetric about it. We introduce a new variable:

$$
x = u - 1. \tag{19}
$$

Table 1
Comparison of approximate periods with exact period

A	T_e	T_1/T_e	T_2/T_e	T_3/T_e
1.42	15.0844	0.581955	0.729206	0.875313
1.45	11.2132	0.737748	0.889001	0.978779
1.5	9.22366	0.821562	0.949312	0.994180
1.7	6.35285	0.915341	0.989117	0.999007
2	4.68568	0.948183	0.996020	0.999621
5	1.52860	0.975637	0.999140	0.999912
10	0.747096	0.977660	0.999278	0.999926
50	0.148369	0.978253	0.999317	0.999929
100	0.0741684	0.978271	0.999318	0.999929
1000	0.00741630	0.978277	0.999318	0.999929

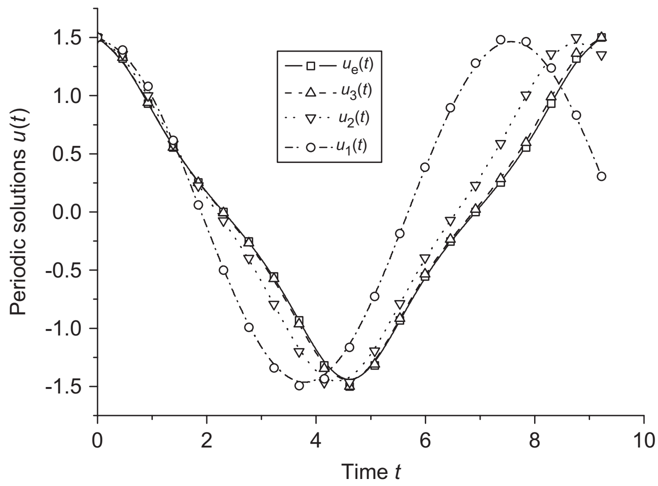

Fig. 1. Comparison of approximate periodic solutions with exact periodic solution for $A = 1.5$.

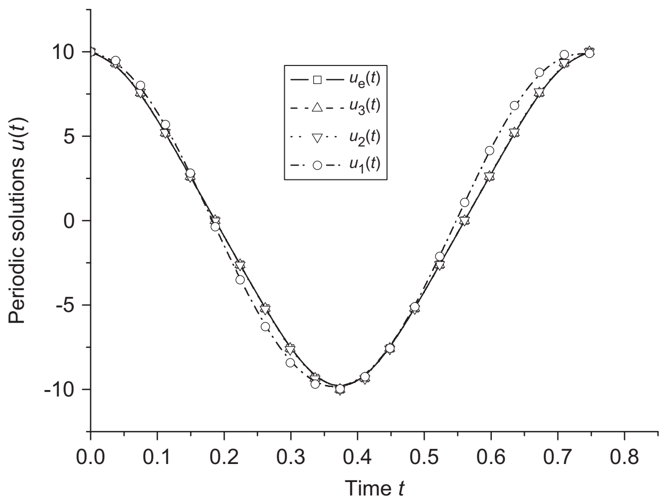

Fig. 2. Comparison of approximate periodic solutions with exact periodic solution for $A = 10$.

Substituting Eq. (19) into Eq. (11) yields

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2x + 3x^2 + x^3 = 0, \quad x(0) = \hat{A}, \quad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0, \tag{20}$$

where $\hat{A} = A - 1$. The corresponding potential energy of the system is

$$V(x) = x^2 + x^3 + x^4/4. \tag{21}$$

Using Eqs. (7) and (21), we can express $\hat{B}(\hat{B} > 0)$ in terms of $\hat{A}$:

$$\hat{B} = 1 - \sqrt{1 - 2\hat{A} - \hat{A}^2}. \tag{22}$$

Based on Eqs. (8) and (9), the introduced odd nonlinear oscillating systems are:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \mathbf{K}(x, \alpha) = 0, \quad x(0) = H, \quad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0, \tag{23}$$

where

$$\mathbf{K}(x,\alpha) = \begin{cases} 2x + 3\alpha x^2 + x^3 & \text{if } x \geqslant 0, \\ 2x - 3\alpha x^2 + x^3 & \text{if } x < 0 \end{cases}$$

with $\alpha = \pm 1$

The Fourier series expansions of $\mathbf{K}(H\cos\tau, \alpha)$ and $\mathbf{K}_x(H\cos\tau, \alpha)$ are given by Eqs. (3) and (5), respectively, where

$$\begin{aligned} &a_1 = 2H + 3H^3/4 + 8\alpha H^2/\pi, \quad a_3 = H^3/4 + 8\alpha H^2/(5\pi), \quad b_0 = 4 + 3H^2 + 24\alpha H/\pi, \\ &b_2 = 3H^2/2 + 8\alpha H/\pi, \quad b_4 = -8\alpha H/(5\pi), \quad b_6 = 24\alpha H/(35\pi). \end{aligned} \tag{24}$$

From Eqs. (2), (4) and (24), we obtain the first two analytical approximate formulas for the period and periodic solution:

$$\begin{aligned} &T_1^\alpha(H) = \frac{2\pi}{\sqrt{\Omega_1^\alpha(H)}}, \quad \Omega_1^\alpha(H) = \frac{1}{4\pi}(8\pi + 3\pi H^2 + 32\alpha H), \\ &x_1^\alpha(t) = H\cos\tau, \quad \tau = \sqrt{\Omega_1^\alpha(H)}t \end{aligned} \tag{25}$$

and

$$\begin{aligned} &T_2^\alpha(H) = \frac{2\pi}{\sqrt{\Omega_2^\alpha(H)}}, \\ &x_2^\alpha(t) = X^\alpha(H)\cos\tau + Y^\alpha(H)\cos 3\tau, \quad \tau = \sqrt{\Omega_2^\alpha(H)}t, \end{aligned} \tag{26}$$

where

$$\begin{aligned} \Omega_2^\alpha(H) = &\big[175\pi^2(512 + 384H^2 + 69H^4) + 480\alpha\pi H(1480 + 541H^2) \\ &+ 1386496\alpha^2H^2\big]/[20\pi\Phi^\alpha(H)], \\ X^\alpha(H) = &[35H\pi(64 + 23H^2) + 8576\alpha H^2\big]/\Phi^\alpha(H), \quad Y^\alpha(H) = 7H^2(5H\pi + 32\alpha)/\Phi^\alpha(H), \\ \Phi^\alpha(H) = &40(56\pi + 21H^2\pi + 220\alpha H). \end{aligned} \tag{27}$$

Table 2
Comparison of approximate periods with exact period

A	T_e	T_1/T_e	T_2/T_e
1.05	4.45169	0.999970	1.00000
1.10	4.48053	0.999866	1.00001
1.15	4.53484	0.999649	1.00003
1.20	4.62391	0.999230	1.00007
1.25	4.76522	0.998401	1.00014
1.30	4.99674	0.996583	1.00032
1.35	5.42749	0.991564	1.00082
1.40	6.75637	0.961218	1.00269
1.41	7.92344	0.916475	1.00051
1.412	8.55534	0.886769	0.995337

To save space, the corresponding third analytical approximation is omitted. The exact period for Eq. (20) is

$$T_e\left(\hat{A}\right) = \int_0^{\pi/2} \frac{2\,dt}{\sqrt{2+\hat{A}(3-\cos 2t+2\sin t)\Big/\left(\cos t/2+\sin t/2\right)^2+\hat{A}^2\left(1+\sin^2 t\right)\Big/2}} + \int_0^{\pi/2} \frac{2dt}{\sqrt{2-\hat{B}(3-\cos 2t+2\sin t)\Big/\left(\cos t/2+\sin t/2\right)^2+\hat{B}^2\left(1+\sin^2 t\right)\Big/2}}, \tag{28}$$

where $\hat{B}$ is given, in terms of $\hat{A}$, in Eq. (22).

By setting $\alpha = 1$, $H = \hat{A}$ and $\alpha = -1$, $H = \hat{B}$, respectively, in Eqs. (25)–(27) and applying the relation in Eq. (10), we can obtain the first two analytical approximate period and periodic solutions T_n and $x_n(t)$ ($n = 1,2$). Based on Eq. (19), the analytical approximate periodic solutions to Eq. (11) are given by

$$u_n(t) = x_n(t) + 1, \quad n = 1,2 \tag{29}$$

and the corresponding approximate periods are the same as T_n ($n = 1,2$).

For comparison, the exact period T_e obtained by integrating Eq. (28) and the approximate periods T_1 and T_2 computed, respectively, by Eqs. (10a), (25) and (26) are listed in Table 2. Furthermore, we have

$$\lim_{A\to+1}\frac{T_1}{T_e} = \lim_{A\to+1}\frac{T_2}{T_e} = 1.00000. \tag{30}$$

Note again that Eq. (11) has a homoclinic orbit with period $+\infty$ for $A = \sqrt{2}$.

From Table 2 and Eq. (30), we can conclude that T_2 gives excellent approximate periods for the oscillation amplitude $1<A<\sqrt{2}$.

The exact periodic solutions $u_e(t)$ achieved by integrating Eq. (11) and the analytical approximate periodic solutions $u_1(t)$ and $u_2(t)$computed by Eqs. (10b), (25) and (26) are plotted in Figs. 3 and 4 for the time in one according period. These figures correspond to, respectively, two different amplitudes of oscillation $A = 1.1$ and 1.4.

Figs. 3 and 4 show that the second analytical approximations provide the most excellent solutions with respect to the exact periodic solutions for oscillation amplitudes $1<A<\sqrt{2}$. The proposed first analytical approximate solution is generally acceptable for oscillation amplitude $A>1$ and near 1.

For the case of $0<A<1$, the oscillation is the same as that with initial conditions $u(0) = \tilde{A} = \sqrt{2-A^2}$, $(du/dt)(0) = 0$, since $V(\tilde{A}) = V(A)$. Note that for $0<A<1$, one has $1<\tilde{A}<\sqrt{2}$. Furthermore, for the case of

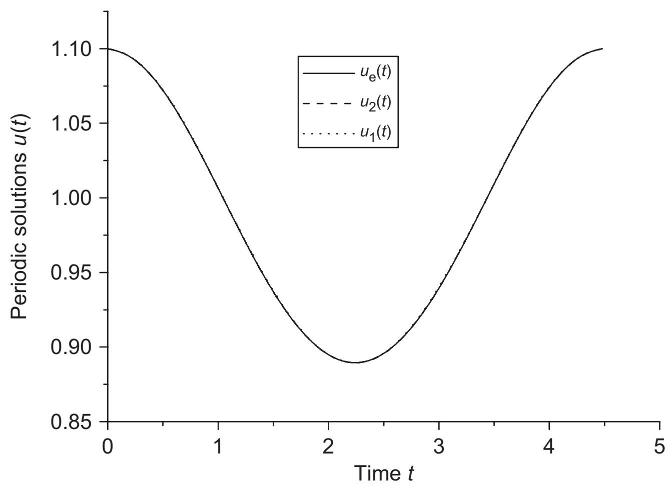

Fig. 3. Comparison of approximate periodic solutions with exact periodic solution for $A = 1.1$.

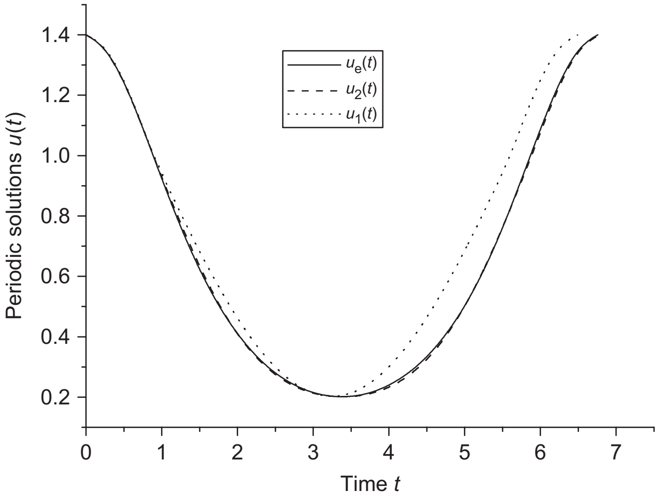

Fig. 4. Comparison of approximate periodic solutions with exact periodic solution for $A = 1.4$.

$1<\tilde{A}<\sqrt{2}$, the corresponding analytical approximate period and periodic solution have been established in above paragraphs. We can then get the analytical approximate period and periodic solution for $0<A<1$.

Because the periodic motion around the equilibrium point $u=-1$ is similar to that around the equilibrium point $u=+1$. The results above may easily be transformed to those for oscillation amplitudes $-\sqrt{2}<A<-1$ and $-1<A<0$. The details are omitted for saving space.

In summary, accurate analytical approximate solutions to the double-well Duffing oscillator have been presented. The solutions are obtained by combining Newton's method with the harmonic balance method. The procedure yields rapid convergence with respect to exact solution. The results are valid for small as well as large oscillation amplitudes.

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